ISRAEL JOURNAL OF MATHEMATICS, Vol. 30, Nos. 1–2, 1978

A BANACH SPACE WITH FEW OPERATORS

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ABSTRACT

Assuming the axiom (of set theory) $V = L$ (explained below), we construct a Banach space with density character $\aleph_1$ such that every (linear bounded) operator T from B to B has the form $a\mathbf{I} + T_1$, where $\mathbf{I}$ is the identity, and T_1 has a separable range. The axiom $V = L$ means that all the sets in the universe are in the class L of sets constructible from ordinals; in a sense this is the minimal universe. In fact, we make use of just one consequence of this axiom, $\diamondsuit_{\aleph_1}$ proved by Jensen, which is widely used by mathematical logicians.

NOTATION. Let i, j, α, β, γ, δ be ordinals, ω the first infinite ordinal, ω_1 the first uncountable ordinal. Let k, l, m, n, p be natural numbers, and let a, b, c, d be reals, and x, y, z elements of a (vector, or norm, or Banach) space.

THE MAIN THEOREM. *Assume the axiom $V = L$ holds. Then there is a Banach space $\bar{Z}$, and an element of the space z_i $(i < \omega_1)$, such that:*

(1) $\mathrm{span}\{z_i : i < \omega_1\}$ *is dense in $\bar{Z}$, $\|z_i\| = 1$, and there are projections $P_\alpha(\alpha < \omega_1)$ of norm 1 of $\bar{Z}$ into itself, $P_\beta(z_i) = 0$ for $i \geqq \beta$, $P_\beta(z_i) = z_i$ for $i < \beta$. So the density character of $\bar{Z}$ is ω_1 and it has a basis $\{z_i : i < \omega_1\}$.*

(2) *If $T : B \to B$ is (linear, bounded) operator, then for some real a, $Tz_i = az_i$ for all but countably many i's. So $T - a\mathbf{I}$ is an operator with a separable range.*

REMARKS. (1) We can prove similar theorems for higher cardinals, i.e., if $\diamondsuit(\{\delta < \lambda^+ : \mathrm{cf}\delta = \lambda\})$, we can construct a space with density character λ^+ such that every operator T of the space is $a\mathbf{I} + T_1$; T_1 has range with density character λ.

(2) We can choose our space so that for every uncountable set of z_i's, there is a countable set which generates an l_∞-Banach space and an l_1-Banach space.

The construction

STAGE A. Let $\{z_i : i < \omega_1\}$ generate freely a vector space H over Q (the rationals). For a set I of ordinals let H_I and also $H(I)$ denote $\mathrm{span}\{z_i : i \in I\}$

Received February 1, 1977

(= the subvector space spanned by z_i, $i \in I$). As an ordinal i is $\{j : j < i\}$, H_i is the vector space spanned by $\{z_j : j < i\}$.

Let I_i^m ($m < \omega$, $i < \omega_1$) be finite subsets of i, increasing with m, and $i = \bigcup_m I_i^m$. For subsets $A_1, A_2, \cdots$ of H, $\langle A_1, A_2, \cdots \rangle_H$ is the span of $A_1 \cup A_2 \cup \cdots$. We usually omit H and write y instead of $\{y\}$.

STAGE B. A subset I of ω_1 is called closed if for each limit ordinal $i < \omega_1$ which satisfies $(\forall j < i)\ (\exists \alpha)\ (j < \alpha < i \wedge \alpha \in I)$ belong to I. I is unbounded if $(\forall i < \omega_1)\ (\exists j < \omega_1)\ (i < j \wedge j \in I)$. A set of $I \subseteq \omega_1$ is called *stationary* if it has a non-empty intersection with every closed unbounded subset of ω_1.

STAGE C. By Jensen [1], if $V = L$ then there are sets D_i, functions $f_i\, (i < \omega_1)$ and $r_i \in \{0, 1\}$ such that

(i) f_i is a two-place function from H_i into the reals, D_i a subset of i.

(ii) For every subset D of ω_1 and two-place function from H into the reals, and $r \in \{0, 1\}$, $\{i < \omega_1 : D \cap i = D_i, f \mid H_i = f_i, r_i = r\}$ is a stationary subset of ω_1.

From now on f_i are as above.

STAGE D. In a norm space Z, for $z \in Z$, $X \subseteq Z$, we say z is *good* over X if $(\forall x \in X)\ \| z + x \| \geqq \| z \|, \| x \|$ and $\| z \| = 1$.

If $z_0, \cdots, z_k \in Z$, $X \subseteq Z$ we say $(z_0, \cdots, z_k)$ is good over X if $\| z_l \| = 1$ and for any reals a_l and $x \in X$

$$\left\| \sum_{l=0}^{k} a_l z_l + x \right\| \geqq \left\| \sum_{l=0}^{k} a_l z_l \right\|, \ \| x \| .$$

Note that (a) (z_0) is good over X iff z_0 is good over X; (b) if $(z_0, \cdots, z_k)$ is good over X then so is every sequence from $\langle z_0, \cdots, z_k \rangle$.

STAGE E. Suppose Y, Z are norm spaces, $Y \cap Z = X$, and let W be a vector space such that Y, Z are subspaces of it, and $W = Y + Z$ (as vector spaces). We can define a norm on W which extends the norms on Y and Z, and get a norm space, as follows:

$$\| w \| = \inf\{\| y \| + \| z \| : y \in Y;\ z \in Z, w = y + z\}.$$

In this case the unit ball of W is the convex hull of the unit balls of Y and Z. We call this N_1-amalgamation. Note that

(a) if $y \in Y$ is good over X, it will be good over Z; and

(b) if also $z \in Z$ is good over X then $\| y + z \| = 2$.

STAGE F. Suppose that in Stage E,

$$Y = \langle X, y_0, \cdots, y_k \rangle, \quad Z = \langle X, z_0, \cdots, z_l \rangle,$$

$(y_0, \cdots, y_k)$, $(z_0, \cdots, z_l)$ are good over X. Then there is another way to define a norm on W extending the norms on Y and Z: for $x \in X$

$$\left\| \sum_{n=0}^{k} b_n y_n + \sum_{n=0}^{l} c_n z_n + x \right\| = \max \left\{ \left\| \sum_{n=0}^{k} b_n y_n + x \right\|, \quad \left\| \sum_{n=0}^{l} c_n z_n + x \right\| \right\}.$$

We call this N_∞-amalgamation (unlike N_1-amalgamation, it apparently does not depend only on Y and Z, but also on span$\{y_0, \cdots, y_k\}$ and span$\{z_0, \cdots, z_k\}$).

Note that

(a) $(z_0, \cdots, z_l)$ is good over Y,

(b) for $n \leqq l$, $m \leqq k$, $z_n + y_n$ is good over X and in particular $\| z_n + y_n \| = 1$,

(c) if $l(1) < l$, and we first amalgamate X, $\langle X, y_0, \cdots, y_k \rangle$, $\langle X, z_0, \cdots, z_{l(1)} \rangle$ in the above-mentioned way and then amalgamate $X' = \langle X, z_0, \cdots, z_{l(1)} \rangle$, $\langle X', y_0, \cdots, y_k \rangle$, $\langle X', z_{l(1)+1}, \cdots, z_l \rangle$, we get the same norm.

STAGE G. We shall define by induction on $i < \omega_1$ norm spaces Z_i, increasing with i, such that Z_i as a vector space is H_i, and for some i's, infinite sets $S_i \subset \omega$ and elements $y_i^m, y_i^m \in H_i$ (for $m < \omega$) when $r_i = 0$, and $y_{i,l}^m$ ($m < \omega$, $1 \leqq l \leqq p(m, i)$) when $r_i = 1$, such that (not distinguishing strictly between subspaces of H_i and of Z_i)

(*) if $\gamma \leqq \alpha_0 < \alpha_1 < \cdots < \alpha_k \leqq i$, $\omega \leqq i$, k a natural number, $r_\gamma = 0$, y_γ^0 is defined, *then* for infinitely many $m \in S_\gamma$

(I) the amalgamation of the triple

$$H(I_\gamma^m), \ \langle H(I_\gamma^m), y_\gamma^m \rangle, \ \langle H(I_\gamma^m), z_{\alpha_0}, \cdots, z_{\alpha_k} \rangle$$

is by the N_∞-amalgamation, i.e., for $x \in H(I_i^m)$

$$\left\| a y_i^m + \sum_{l=0}^{k} b_l z_{\alpha_l} + x \right\| = \max \left\{ \left\| a y_i^m + x \right\|, \quad \left\| \sum_{l=0}^{k} b_l z_{\alpha_l} + x \right\| \right\};$$

(II) the amalgamation of the triple

$$\langle H(I_\gamma^m), y_\gamma^m \rangle, \quad \langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m \rangle, \quad \langle H(I_\gamma^m), y_\gamma^m, z_{\alpha_0}, \cdots, z_{\alpha_k} \rangle$$

is by the N_1-amalgamation.

So in particular

(*′) z_γ is good over H_γ, and if $\gamma \leqq \alpha_0 \leqq \alpha_1 \cdots \leqq \alpha_k$ then $(z_{\alpha_0}, \cdots, z_{\alpha_k})$ is good over H_γ.

We also demand

(**) if $\gamma \leqq \alpha_0 < \alpha_1 < \cdots < \alpha_k \leqq i$, $\omega \leqq i$, k a natural number, $r_\gamma = 1$, *then* for infinitely many $m < \omega$ the amalgamation of the triple

$$H(I^m_\gamma), \langle H(I^m_\gamma), y^m_{i,1}, \cdots, y^m_{i,p(m,\gamma)}\rangle, \langle H(I^m_\gamma), z_{\alpha_0}, \cdots, z_{\alpha_k}\rangle$$

is by N_∞-amalgamation.

For i limit $Z_i = \bigcup_{j<i} Z_j$, for $i < \omega$, $\|\Sigma_{l<i} a_l z_l\| = \max_{l<\omega} |a_l|$.

STAGE H. Now we do the induction step, so we suppose the norm on H_i is defined, $i \geqq \omega$, and we call the norm space Z_i. In this stage we shall define y^m_i, y^m_i ($m < \omega$) and S_i, and in the next stage we shall define the norm on H_{i+1}. Remember that f_i is a two place function from H_i to R given by the Jensen diamond (see Stage B).

If there is a (bounded) operator T on $\bar{Z}_i$ such that for every $x, y, \in H_i$, $f_i(x,y) = \|Tx - y\|$, it is unique, and we call it T_i.

If T_i is not defined we do not define S_i, y^m_i, y^m_i. So suppose T_i is defined.

(a) If Y is a Banach space, T an operator on Y, $H \subseteq Y$ a subspace, then let

$$c(H,T,Y) = \sup\{d(Ty, \langle H, y\rangle) : y \in Y, y \text{ good over } H\},$$

where $d(y_1, H_1)$ is the distance between y_1 and H_1, i.e., $\inf\{\|y_1 - x\| : x \in H_1\}$, and let

$$c_\varepsilon(H,T,Y) = \sup\{d(Ty,H) : d(Ty,\langle H,y\rangle) \geqq c(H,T,Y) - \varepsilon$$
$$\text{and } y \text{ is good over } H\}.$$

Note that $c(H,T,Y) \leqq \|T\|$ and it decreases with H.

Now if $r_i = 0$, choose y^m_i, y^m_i in H_i such that:

(b) (i) $d(Ty^m_i, \langle H(I^m_i), y^m_i\rangle) \geqq c(H(I^m_i), T_i, \bar{Z}_i) - 1/m$,

(ii) y^m_i is good over $H(I^m_i)$,

(iii) $d(Ty^m_\gamma, H(I^m_\gamma)) \geqq c_{1/m}(H(I^m_i), T_i, \bar{Z}_i) - 1/m$,

(iv) $\|Ty^m_i - y^m_i\| < 1/m$.

Clearly $c_{1/m}(H(I^m_i), T_i, \bar{Z}_i)$ is a real number of absolute value $< \|T\|$, hence there is an infinite set $S_i \subseteq \omega$ such that

(c) for $k < m \neq n$ in S_i, $1/k > |c_{1/m}(H(I^m_i), T_i, \bar{Z}_i) - c_{1/n}(H(I^n_i), T_i, \bar{Z}_i)|$.

(d) If $r_i = 1$ choose a $p = p(m,i) < \omega$ and $y^m_{i,l} \in \{z_\alpha : \max I^m_i < \alpha < i, \alpha \in D_i\}$ such that:

(i) for every $x \in H(I^m_i)$,

$$\left\| \sum_{l=1}^{p(m)} a_l y^m_{i,l} + x \right\| = \sup \|a_l y^m_{i,l} + x\|$$

(notice each $y^m_{i,l}$ is good over $H(I^m_i)$),

(ii) if among the p's satisfying (i) there is a maximal one, this will be our p; otherwise choose $p = m$.

STAGE I. Now we have to define the norm on H_{i+1} (after we have defined it on H_i), and define, if necessary, y_i^m, y_i^m $(m < \omega)$ or $y_{i,l}^m$.

We have to satisfy the requirements $(*)$ and $(**)$ from Stage G; when $\alpha_k < i$ they are satisfied by the induction hypothesis. Clearly there are only countably many appropriate requirements, so we can find a list of them of length ω, each appearing infinitely many times.

Let $\{\beta_n : n < \omega\}$ be a list of $i = \{j : j < i\}$. Now we define by induction on $n < \omega$ a finite set $J_n \subseteq i$, and a norm space Z_i^m which as a vector space is $H(J_n \cup \{i\})$ (we shall not distinguish) such that

(i) $J_n \subseteq J_{n+1}$,

(ii) Z_i^m is a subspace of Z_i^{n+1},

(iii) $i = \bigcup_{n<\omega} J_n$,

(iv) in Z_i^m, z_i is good over $H(J_n)$.

For $n = 0$ let H_0 be the empty set, and the norm Z_i^0 is $\|az_i\| = |a|$.

Suppose we have defined Z_i^n for n, and let us define Z_i^{n+1}. Let $\langle k, \gamma, \alpha_0, \cdots, \alpha_{k-1}\rangle$ be the n-th in the list of cases of $(*)$ and $(**)$ from Stage G. Assume for now that $r_\gamma = 0$ (the case $r_\gamma = 1$ is just simpler). If $\{\alpha_0, \cdots, \alpha_{k-1}\} \not\subseteq J_n$, we let $J_{n+1} = J_n \cup \{\beta_n\}$, and we define the norm of Z_i^{n+1} by N_1-amalgamation of $H(J_n)$, Z_i^n, $H(J_{n+1})$ (see Stage E).

Now if $\{\alpha_0, \cdots, \alpha_{k-1}\} \subseteq J_n$, let $J_n - \gamma = \{\beta_0, \cdots, \beta_l\}$ (as $\gamma \leqq \alpha_0 < \cdots$ necessarily $\{\alpha_0, \cdots, \alpha_{k-1}\} \subseteq \{\beta_0, \cdots, \beta_l\}$). By the induction hypothesis, $(*)$ of Stage G holds for $\gamma \leqq \beta_0 \leqq \cdots \leqq \beta_l$ hence there is an $m \in S_\gamma$ satisfying

(1) $J_n \cap \gamma \subseteq I_\gamma^m$ (possible as $(*)$ says "for infinitely many m's" and $\gamma = \bigcup_m I_\gamma^m$, I_γ^m increase with m, and J_n is finite),

(2) the amalgamation of the triple

$$H(I_\gamma^m), \quad \langle H(I_\gamma^m), y_\gamma^m\rangle, \quad \langle H(I_\gamma^m), \quad z_{\beta_0}, \cdots\rangle$$

is an N_∞-amalgamation,

(3) the amalgamation of the triple

$$\langle H(I_\gamma^m), y_\gamma^m\rangle, \quad \langle H(I_\gamma^m), y_\gamma^m, y_\gamma^m\rangle, \quad \langle H(I_\gamma^m), y_\gamma^m, z_{\beta_0}, \cdots\rangle$$

is an N_1-amalgamation.

We choose a finite J_{n+1} such that $J_n \subseteq J_{n+1} \subseteq i$, $\beta_n \in J_{n+1}$ and $y_\gamma^m, y_\gamma^m \in H(J_{n+1})$ (this is trivial). Now we define Z_i^{n+1} by successive amalgamation.

(α) We make an N_1-amalgamation of the triple $H(J_n)$, Z_i^n, $H(J_n \cup I_\gamma^m)$: z_i is good (in it) over $H(J_n \cup I_\gamma^n)$ by (a) of Stage E.

(β) We make an N_∞-amalgamation of $H(J_n \cup I_\gamma^m)$, $H(J_n \cup I_\gamma^m \cup \{i\})$ (defined in α), and $\langle H(J_n \cup I_\gamma^m), y_\gamma^m\rangle$ (possible as z_i is good over $H(J_n \cup I_\gamma^m)$ by (α) and y_γ^m

is good over $H(J_n \cup I^m_\alpha)$ by the choice of m to satisfy (2) and (a) of Stage F). By (a) of Stage F, z_i is good over $\langle H(J_n \cup I^n_\gamma), y^n_\gamma\rangle$ in the amalgamated space we have just defined.

(γ) We make the N_1-amalgamation of

$$\langle H(J_n \cup I^m_\gamma), y^m_\gamma\rangle,\quad H(J_{n+1}),\quad \langle H(J_n \cup I^m_\gamma \cup \{i\}), y^m_\gamma\rangle$$

(with the norm defined in (β)) and call it Z^{n+1}_i. By (a) of Stage E z_i is good over $H(J_{n+1})$ in Z^{n+1}_i.

It is easy to check that (I) and (II) of (*) hold for $\gamma, \alpha_0, \cdots, \alpha_k$ and m (by (c) of Stage F).

So Z^n_i is defined for every i, and let $Z_{i+1} = \bigcup_{n<\omega} Z^n_i$. Clearly Z_{i+1} as a vector space is H_{i+1} (as $\beta_n \in J_{n+1}$). Each requirement $\gamma \leqq \alpha_0 < \cdots < \alpha_k = i$ appears in our list infinitely many times so for every n big enough $\{\alpha_0, \cdots, \alpha_k\} \subseteq J_n$, so clearly (*) holds for $i+1$.

STAGE J. We have defined Z_i for $i < \omega_1$. Let $Z = \bigcup_{i<\omega_1} Z_i$ (so as a vector space it is H), and $\bar{Z}$, its completion, is the Banach space which exemplifies our theorem.

So let T be an operator on Z, and we shall prove it is as mentioned in the theorem, i.e., for some a, for every large enough i, $Tz_i = az_i$. We define a two place function f from H into R:

$$f(x,y) = \| Tz - y \|.$$

By Stage B

$$I = \{i < \omega_1 : f \restriction H_i = f_i, r_i = 0\}$$

is a stationary subset of ω_1 (see Stage B).

STAGE K. For each finite-dimensional subspace G of Z and $m < \omega$ there is $y^m_G \in Z$ good over G such that

$$d(Ty^m_G, G, y^m_G) \geqq c(G, T, \bar{Z})(1 - 1/m) \quad d(T^m_G y, H) \geqq c_{1/m}(G, T, \bar{Z}) - 1/m.$$

For each $x \in Z$ there is $i(x) < \omega_1$ such that $x, Tx \in \bar{Z}_{i(x)}$. Now for each $\alpha < \omega_1$, $A_\alpha = \{i(x) : x \in H_\alpha$ or $x = y^m_G$ for some finite-dimensional $G \subseteq H_m, m < \omega\}$ is countable, hence $i(\alpha) = \sup A_\alpha < \omega_1$. Now $A = \{j < \omega_1 : (\forall \alpha < j) i(\alpha) < j\}$ is a closed unbounded subset of ω_1 (closed-trivially by the definition, unbounded because $i(\alpha)$ increases with α, so if $j_0 = j, j_{n+1} = i(j_n)$, then $j_0 \leqq \bigcup_n j_n < \omega_1$ and $\bigcup_n j_n$ is in this set). As I is stationary (see Stage B for definition, and Stage J for the fact) there is $\gamma \in A \cap I$ (I from Stage J). Clearly T maps Z_γ into $\bar{Z}_\gamma$, hence it maps $\bar{Z}_\gamma$ into $\bar{Z}_\gamma$, and

$$c(H(I^m_\gamma), T, \bar{Z}) = c(H(I^m_\gamma), T_\gamma, \bar{Z}_\gamma) \quad \text{and}$$

$$c_{1/m}(H(I^m_\gamma), T, \bar{Z}) = c_{1/m}(H(I^m_\gamma), T_\gamma, \bar{Z}_\gamma)$$

(as $\gamma \in A$) and $T_\gamma = T \mid Z_\gamma$ (as $\gamma \in I$).

STAGE L. Now we shall prove that for every $i > \gamma$, $Tz_i \in \langle Z_\gamma, z_i \rangle$ (γ is as chosen at the end of Stage K, and will remain fixed).

For this it suffices to prove that for any real $\varepsilon > 0$, $d(Tz_i, \langle H(I^m_\gamma), z_i \rangle) \leqq 5\varepsilon$ for some $m < \omega$. So let $\varepsilon > 0$ be given. Now Tz_i is in the closure of $Z = \operatorname{span}\{z_\alpha : \alpha < \omega_1\}$, so for some $l(0) < \omega$ and $a_l \in R$, and distinct $\beta(l) < \omega_1$ (for $l \leqq l(0)$):

(a) $\| Tz_i - \sum_{l \leq l(0)} a_l z_{\beta(l)} \| < \varepsilon$.

So we can choose $k < \omega$, and $\alpha_0 < \cdots < \alpha_k < \omega_1$, $\gamma \leqq \alpha_0$ such that

$$\{i, \beta(0), \cdots, \quad \beta(l(0))\} - \gamma \subseteq \{\alpha_0, \cdots, \alpha_k\}.$$

Now by (*) (from Stage G), for infinitely many $m \in S_\gamma$, I and II from (*) hold (for our $k, \gamma, \alpha_0, \cdots, \alpha_k$). So we can choose some m for which $\{\beta(0), \cdots, \beta(l(0))\} \cap \gamma \subseteq I^m_\gamma$ and $1/m < \varepsilon$. Clearly

(b) $z_i =^{\text{def}} \sum_{l \leq l(0)} a_l z_{\beta(l)} \in H(I^m_\gamma \cup \{\alpha_0, \cdots, \alpha_k\})$

and by I of (*) and Stage F

(c) $z_\gamma + y^m_\gamma$ is good over H^m_i.

Now we shall write a series of inequalities which will prove $d(Tz_i, \langle H(I^m_i), z_i \rangle) \leqq 5\varepsilon$; for notational convenience let x range over $H(I^m_\gamma)$, and a, b range over R.

(d) $c(H(I^m_\gamma), T_i, \bar{Z}_i) =$ [as $\gamma \in A$, see Stage K]

$c(H(I^m_\gamma), T, \bar{Z}) \geqq$ [by c's definition, and (c) above]

$d(T(z_i + y^m_\gamma), \langle H^m_\gamma, z_i + y^m_\gamma \rangle) \geqq$ [by d's definition]

$\inf_{a,x} \| T(z_i + y^m_\gamma) + a(z_i + y^m_\gamma) + x \| \geqq$ [as $\| Tz_i - z_i \| < \varepsilon$, $Ty^m_\gamma = T_i y^m_\gamma$ and $\| T_1 y^m_\gamma - y^m_\gamma \| \leqq 1/m$ as mentioned in (b) of stage H]

$\inf_{a,x} \| z_i + y^m_\gamma + az_i + ay^m_\gamma + x \| - 1/m - \varepsilon \geqq$ [by II of (*)]

$\inf_{a,b,x,x_1} (\| y^m_\gamma + ay^m_\gamma + x + (by^m_\gamma + x_1) \| +$

$$+\|z_i + az_i - (by^m_\gamma + x_1)\| - 1/m - \varepsilon) = \inf_{a,b,x_1,x_2} (\|y^m_\gamma + ay^m_\gamma + by^m_\gamma + x_1\| +$$

$$+\|z_i + az_i - by^m_\gamma + x_2\| - 1/m - \varepsilon) \geqq$$

$$\inf_{a,b,x_1,x_2} \|y^m_\gamma + ay^m_\gamma + by^m_\gamma + x_1\| + \inf_{a,b,x_1,x_2} \|z_i + az_i - by^m_\gamma + x_2\| - 1/m - \varepsilon \geqq$$

[as $\|Ty^m_\gamma - y^m_\gamma\| < 1/m$, $\|Tz_i - z_i\| < \varepsilon$]

$$\inf_{a,b,x_1} \|Ty^m_\gamma + ay^m_\gamma + by^m_\gamma + x_1\|$$

$$- 1/m + \inf_{a,b,x_2} \|Tz_i + az_i - by^m_\gamma + x_2\| - \varepsilon - 1/m - \varepsilon \geqq \quad \text{[by } d\text{'s definition]}$$

$$d(Ty^m_\gamma, \langle H^m_\gamma, y^m_\gamma\rangle) + \inf_{a,b,x} \|Tz_i + az_i - by^m_\gamma + x\| - 2/m - 2\varepsilon \geqq$$

[by (b) of Stage H]

$$c(H(I^m_\gamma), T_i, \bar{Z}_i) - 1/m + \inf_{a,b,x} \|Tz_i + az_i - by^m_i + x\| - 2/m - 2\varepsilon.$$

Comparing the first and last elements we see that

(e) $\inf_{a,b,x} \|Tz_i + az_i - by^m_\gamma + x| \leqq 3/m + 2\varepsilon$.

Now by the choice of m

(f) $1/m < \varepsilon$.

Combining we get $d(Tz_i, \langle H^{m+1}_\gamma, z_i\rangle) \leqq d(Tz_i, \langle H^m_\gamma, y^m_\gamma\rangle \leqq 3/m + 2\varepsilon < 5\varepsilon$.

STAGE M. For each $\beta < \omega_1$ we define an operator P_β on $\bar{Z} : P_\beta(z_i) = 0$ for $i \geqq \beta$, and $P_\beta(z_i) = z_i$ for $i < \beta$. It is easy to check that:

(a) P_β is well defined and is a projection with norm 1 onto Z_β;

(b) for $\beta < \alpha$, $P_\beta P_\alpha = P_\alpha P_\beta = P_\beta$.

(c) If $P_\alpha(x) \neq 0$, α limit, then for some $\beta < \alpha$, $P_\beta(x) \neq 0$.

STAGE N. Let T, γ be as in Stage L. So for every $i \geqq \gamma$, $Tz_i \in \langle \bar{Z}, z_i\rangle$, so $Tz_i = d_i z_i + x^0_i$, $x^0_i \in \bar{Z}_\gamma$.

We shall prove that for some $\delta, \gamma \leqq \delta < \omega_1$, and for every $i \geqq \delta$, $x^0_i = 0$. Suppose not, so $A_1 = \{i < \omega_1 : i \geqq \gamma, \|x^0_i\| \neq 0\}$ is uncountable. For each $i \in A_1$ choose a minimal $\beta_i \leqq \gamma$ such that $P_{\beta_i}(x^0_i) \neq 0$ (it exists as $P_\gamma(x^0_i) = x^0_i$, because $x^0_i \in \bar{Z}_\gamma$). By (c) of Stage M β_i is a successor ordinal, so for some $\beta < \gamma$, $A_2 = \{i \in A_1 : \beta_i = \beta + 1\}$ is uncountable. So for each $i \in A_2$, for some real $d^1_i \neq 0$, $P_\beta(x^0_i) = d^1_i z_\beta$. So for some $a > 0$ and $s \in \{1, -1\}$, $A_3 = \{i \in A_2 : sd^1_i > a\}$ is uncountable. So for each $i \in A_3$, $P_\beta T z_i = d^1_i x_\beta$, $sd^1_i > a$.

By Stage B, $I' = \{i < \omega_1 : r_i = 1, f \mid H_i = f_i, A_3 \cap i = D_i\}$ is a stationary subset of ω_1. Let

$$A = \{i < \omega_1 : i \text{ is limit, } i > \gamma, \text{ and } A_3 \cap i \text{ is unbounded below } i, \text{ and in (d) of Stage H, if we ask } y^m_{i,l} \text{ in } \{z_\alpha : \max I^m_i < \alpha,\ \alpha \in A_3\} \text{ the value of } p = p(m,i) \text{ does not change}\}.$$

As in Stage K, we can prove A is closed and unbounded so $I \cap A \neq \emptyset$, and choose in it an element δ. Now for infinitely many $m < \omega, p(m,\delta) \geqq m$. Otherwise choose $m_0 < \omega$ such that

(a) $m \geqq m_0 \Rightarrow p(m,\delta) < m$

and choose $i \in A_2, i > \delta$. By (**) of Stage G, for some $m > m_0$, $H(I^m_\delta)$, $\langle H(I^m_\delta), z_i\rangle$, $\langle H(I^m_\delta), y^m_{\delta,1}, \cdots, y^m_{\delta,p(m)}\rangle$ have N_∞-amalgamation. Now checking (b) of Stage H, we see that z_i was an appropriate candidate for being $y^m_{\delta,p(m,\delta)+1}$ hence $p(m,\delta) = m$, contradiction.

So for m, l, $y^m_{\delta,l} \in \{z_\alpha : \alpha \in A_2\}$, hence $P_\beta T y^m_{\delta,l} \in \{sbx_\beta : b > a\}$. Now for every m, (see (**) of Stage G)

$$\left\| \sum_{l=1}^{p(m,\delta)} y^m_{\delta,l} \right\| = \max_l \| y^m_{\delta,l} \| = 1,$$

$$\begin{aligned}
\left\| T\left(\sum_{l=1}^{p(m,\delta)} y^m_{\delta,l}\right)\right\| &\geqq \left\| P_{\beta+1} T\left(\sum_{l=1}^{p(m,\delta)} y^m_{\delta,l}\right)\right\| && [\text{as } \|P_{\beta+1}\| = 1 \text{ by Stage M}]\\
&= \left\| \sum_{l=1}^{p(m,\delta)} P_{\beta+1} T y^m_{\delta,l} \right\| && [\text{as } P_{\beta+1} T y^m_{\delta,l} \in \{sbx_\beta : b > a\}]\\
&= \sum_{l=1}^{p(m,\delta)} \| P_{\beta+1} T y^m_{\delta,l} \| \\
&\geqq p(m,\delta) a \\
&\geqq ma.
\end{aligned}$$

Hence $\|T\| \geqq ma$, as $a > 0$, m $(m < \omega)$ arbitrarily large, we get a contradiction.

Stage P. (we omit O as a stage). We now want to show that d_i $(i < \omega_1)$ is eventually constant. Otherwise there are distinct reals d^0, d^1 such that

(a) for $l = 1, 2$ and $\alpha < \omega_1$, and $\varepsilon > 0$, there is i, $\alpha < i < \omega_1$, and $|d_i - d^l| < \varepsilon$; w.l.o.g. $d^0 = 0$, $d^1 = 1$ (otherwise, we look at the operator $1/(d^1 - d^0)(T - d^0\mathbf{I})$ ($\mathbf{I}$—the identity operator).

Let $\varepsilon > 0$ be arbitrary, $\varepsilon < 1/100$. Choose $\alpha < \beta < \delta$ $(\geqq \gamma), |d_\alpha| < \varepsilon, |1 - d_\beta| < \varepsilon$. By (*) of Stage G, for $k = 1, \alpha_0 = \alpha, \alpha_1 = \beta, i = \gamma$ we can find $m(1) < m$ in S_γ such that (I) and (II) of (*) holds for m and for $m(1)$ and

$$1/m(1) < \varepsilon, \qquad 12m(1) < m.$$

We now try to get a contradiction to the choice of y^m_γ. We repeat Stage L with z_α for z_i so (b), (c), (d) holds ((a) is trivialized—we know better), but we want to deviate in the middle of (d):

$$c(H(I^m_\gamma), T_i, \bar{Z}_i) \geqq \inf_{a,b,x_1,x_2} (\|y^m_\gamma + ay^m_\gamma + by^m_\gamma + x_1\| + \|Tz_\alpha + az_\alpha - by^m_\gamma + x_2\| - 1/m).$$

So for some a, b, x_1, x_2 we get this infimum up to $1/m$, so

$$c(H(I^m_\gamma), T_i, \bar{Z}_i) + 2/m \geqq \|y^m_\gamma + ay^m_\gamma + by^m_\gamma + x_1\| + \|Tz_\alpha + az_\alpha - by^m_\gamma + x_2\| \geqq$$

[as $\|Ty^m_\gamma - y^m_\gamma\| < 1/m$ and $Tz_\alpha = d_\alpha z_\alpha$]

$$\|Ty^m_\gamma + (a+b)y^m_\gamma + x_1\| + \|d_\alpha z_\alpha + az_\alpha - by^m_\gamma + x_2\| - 1/m = \qquad \text{[by I of } (*)]$$

$$\|Ty^m_\gamma + (a+b)y^m_\gamma + x_2\| + \max\{\|(d_\alpha + a)z_\alpha + x_2\|, \|-by^m_\gamma + x_2\|\} - 1/m$$

[as z_α, y^m_α are good over $H(I^m_\alpha)$]

$$\geqq \|Ty^m_\gamma + (a+b)y^m_\gamma + x_1\| + \max\{|d_\alpha + a|, |b|\} - 1/m$$

$$\geqq d(Ty^m_\gamma, \langle H(I^m_\gamma), y^m_\gamma\rangle) + \max\{|d_\alpha + a|, |b|\} - 1/m$$

$$\geqq c(H(I^m_\gamma), T_\gamma, \bar{Z}_\gamma) + \max\{|d_\alpha + a|, |b|\} - 2/m.$$

We can conclude that

(b) $|b|$, $|d_\alpha + a| < 4/m$,

(c) $\|Ty^m_\gamma + (a+b)y^m_\gamma + x_1\| \leqq d(Ty^m_\gamma, \langle H(I^m_\gamma), y^m_\gamma\rangle) + 4/m$

(for (b) look at the first and last terms in our series of inequalities, for (c), if it fails use this in the passage from the fifth term to the sixth term, and we shall get a contradiction).

Combining (b) and (c) we get

(d) $\|Ty^m_\gamma - d_\alpha y^m_\gamma + x_1\| \leqq d(Ty^m_\gamma, \langle H(I^m_\gamma), y^m_\gamma\rangle) + 12/m$.

Now remember $|d_\alpha| < \varepsilon$, $1/m < \varepsilon$ hence

(e) $\|Ty^m_\gamma + x_1\| \leqq d(Ty^m_\gamma, \langle H(I^m_\gamma), y^m_\gamma\rangle) + 13\varepsilon$.

Similarly, for β instead α, (d) holds, but $|1 - d_\beta| < \varepsilon$ hence for some $x'_1 \in H(I^m_\gamma)$

(f) $\|Ty^m_\gamma - y^m_\gamma + x'_1\| \leqq d(Ty^m_\gamma, \langle H(I^m_\gamma), y^m_\gamma\rangle) + 13\varepsilon$.

By the version of (d) for (β), for $y = y^m_\gamma + z_\beta$ and the choice of y^m_γ in Stage H

(g) $d(Ty, \langle H(I^m_\gamma), y\rangle) > c(H(I^m_\gamma), T_i, \bar{Z}_i) - 1/m\,(1)$.

Now z_β is good over $\langle H(I^m_\gamma), y^m_\gamma, Ty^m_\gamma\rangle$ hence

(h) $d(Ty, H(I^m_\gamma)) = \inf_x \| Ty^m_\gamma + d_\beta z_\beta + x \|$

$$= \inf_{a, x_1, x_2} [\| Ty^m_\gamma - ay^m_\gamma + x_1 \| + \| d_\beta z_\beta - ay^m_\gamma + x_2 \|]$$

$$\geqq d(Ty^m_\gamma, \langle H(I^m_\gamma), y^m_\gamma \rangle) + 1 - \varepsilon$$

[by (e)]

$$\geqq d(Ty^m_\gamma, H(I^m_\gamma)) + 1 - 14\varepsilon$$

So y contradicts the definition of $c_{1/m(1)}(H(I^m_\gamma), \bar{T}_\gamma, \bar{Z}_\gamma)$ and the choice of y^m_γ.

Acknowledgements

I thank L. Tzafriri for many discussions on Banach spaces, and S. Friedland for suggesting the definition

$$\| ay + bz + x \| = \max \{\| ay + x \|, \quad \| bz + x \|\}$$

(see Stage F).

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